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USE OF THE OPERATIONAL METHOD IN SOLVING THE INITIAL CHARACTERISTIC PROBLEM OF THE PLANE THEORY OF ELASTICITY

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T' dimensional Laplace-Carson transform is used to obtain a solution of the Goursat problem for the equation describing plane deformation of an ideally plastic body. The basic relationships in a field formed by circular arcs are investigated as an example.

One of the methods of solving the boundary value problems of a plane flow of an ideally plastic medium consists of linearizing the quasilinear, hyperbolic, partial differential equation with help of the Mikhlin [1] substitution which reduces it to the equation of telegraphy. The linearized equation can be solved using the integral Riemann formula [2, 3], by expanding into a series in metacylindrical functions, or by means of the integral transforms.

1. All functions appearing in this paper are assumed to be Laplace-transformable [5]. We note that this demand does not lead to any restrictions in practice. Let us denote the curvilinear characteristic coordinates in the physical plane by α and β, and in the velocity hodograph plane by α' and β'. As we know, each of the functions X, Y, U, V, R, S, ρ and δ of the plane problem of the theory of ideal plasticity satisfies the equation of telegraphy

$$d^2f/d\zeta d\varkappa + f = 0 \tag{1.1}$$

Here X and Y are the coordinates of the nodal points of the slip-lines in the moving coordinate system, U and V are the coordinates of the nodal points of the velocity hodograph in the moving coordinate system, R and S are the radii of curvature of the α- and β-lines, respectively, while ρ and δ are the radii of curvature of the α'- and β'-lines.

As usual, we shall regard the anticlockwise direction of counting the characteristic coordinates, as positive.

Let Eq. (1.1) be defined in the physical plane, within the rectangle D_0 $(-\infty < \alpha \leqslant 0,\ 0 \leqslant \beta < \infty)$. We denote

$$\gamma = -\alpha = |\alpha|, \quad f(\alpha, \beta) = f(-\gamma, \beta) = \varphi(\gamma, \beta) \tag{1.2}$$

The equation (1.1) defined in the region D $(0 \leqslant \gamma < \infty,\ 0 \leqslant \beta < \infty)$, can be rewritten in the form

$$\frac{\partial^2\varphi(\gamma, \beta)}{\partial\gamma\,\partial\beta} - \varphi(\gamma, \beta) = 0 \tag{1.3}$$

To apply the operational method, we define the following values (where the arrow denotes a passage to the image space):

$$f(|\alpha|, 0) = \varphi(\gamma, 0) = a(\gamma) \to a^*(p) \tag{1.4}$$
$$f(0, \beta) = \varphi(0, \beta) = b(\beta) \to b^*(q), \qquad a(0) = b(0) = c \to c$$

Using the formulas [5] for the transformation of derivatives and the boundary conditions (1.4), we obtain (1.3) in the following operator form:

$$pq\varphi^* (p, q) - pqa^* (p) - pqb^* (q) + pqc - \varphi^* (p, q) = 0 \qquad (1.5)$$

from which we obtain

$$\varphi^* (p, q) = \frac{pq}{pq-1} [a^* (p) + b^* (q) - c] \qquad (1.6)$$

In the space of transformations a product of two functions corresponds to the convolution of the original functions [5], therefore we can write the solution of the equation in the region D_0, with (1.2) taken into account, in the form

$$f(\alpha, \beta) = \int_0^{|\alpha|}\int_0^{\beta} [I_0 (2 \sqrt{\xi\eta}) - 1] [a(|\alpha| - \xi) + b(\beta - \eta) - c] \, d\xi \, d\eta \qquad (1.7)$$

where ($I_\nu (x)$ is a Bessel function with an imaginary argument.

In the velocity hodograph plane the domain of definition of (1.1) is D_1 ($0 \leqslant \alpha' < \infty$; $-\infty < \beta' \leqslant 0$). In this case the substitution

$$\gamma = -\beta' = |\beta'|, \qquad f(\alpha', -\gamma) = \psi(\alpha', \gamma) \qquad (1.8)$$

yields an equation identical with (1.3). Its solution in D_1 has the form (1.7) in which α' and $|\beta'|$ replace $|\alpha|$ and β, respectively, in the right-hand side. We note that in the course of finding the solution in the original space we can apply the inverse Laplace-Carson transform formula [6] directly to (1.6).

2. As an example, we shall consider the field of slip-lines formed by the initial arcs of the radii (Fig. 1)

$$R(\alpha, 0) = R_0 = \text{const}, \qquad S(0, \beta) = S_0 = \text{const}$$

Let the jumps in the tangential velocity component propagating along the slip-lines $\alpha = \alpha_1$ and $\beta = \beta_1$ forming the boundary of the plastic region be, respectively, ΔV and ΔU. Then the velocity field in the velocity hodograph plane (Fig. 2) will be formed by the initial arcs of radii

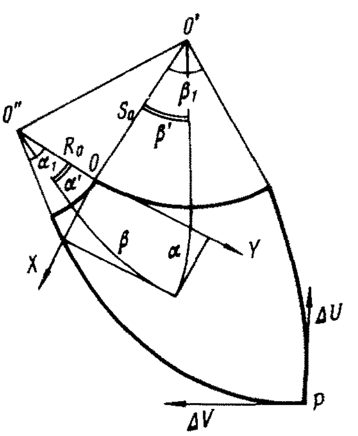

Fig. 1

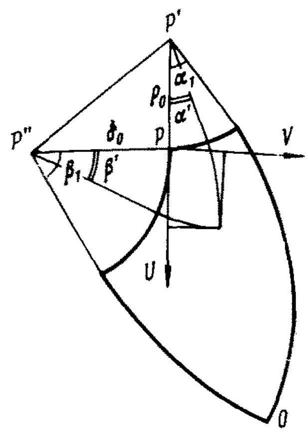

Fig. 2

$$\rho(\alpha', 0) = \Delta U = \rho_0 = \text{const}, \quad \delta(0, \beta') = \Delta V = \delta_0 = \text{const}$$

The boundary conditions for the function V in the moving coordinate system, placed at the point P of the velocity hodograph, have the form

$$V(\alpha', 0) = a(\alpha') = \rho_0 \sin \alpha' \to \frac{\rho_0 p}{p^2+1} = a^*(p) \tag{2.1}$$

$$V(0, |\beta'|) = b(\gamma) = \delta_0 (1 - \cos \gamma) \to \frac{\delta_0}{q^2+1} = b^*(q)$$

Reverting to the formula (1.6), we obtain the solution in the transformation space, and on returning to the original space, we obtain

$$V(\alpha', \beta') = \rho_0\, U_1(2\alpha', 2i\sqrt{\alpha' |\beta'|}) + \delta_0 U_2(2|\beta'|, 2i\sqrt{\alpha' |\beta'|}) \tag{2.2}$$

where ($U_n(w, z)$ is the Lommel function of two variables.

Taking into account the obvious relations connecting the physical plane variables with the hodograph plane variables

$$\alpha' = \alpha - \alpha_1, \qquad \beta' = \beta - \beta_1$$

we can obtain from (2.2) an analytic expression for the velocity vector increment along the β-lines in the physical plane. Similarly, for the function U the boundary conditions

$$U(\alpha', 0) = a(\alpha') = \rho_0 (1 - \cos \alpha') \to \frac{\rho_0}{p^2+1} = a^*(p)$$

$$U(0, \gamma) = b(\gamma) = \delta_0 \sin \gamma \to \frac{\delta_0 q}{q^2+1} = b^*(q)$$

yield an expression for $U(\alpha', \beta')$ which becomes (2.2) when the indices 1 and 2 are interchanged.

Let us now give the intial conditions and the solution for the remaining parameters of the velocity hodograph and slip-line field depicted on Figs. 1 and 2.

The radii of curvature of the α'- and β'-lines in the velocity hodograph plane are

$$\rho(0, \gamma) = \rho_0 + \delta_0 \gamma, \qquad \delta(\alpha', 0) = \delta_0 + \rho_0 \alpha' \tag{2.3}$$

$$\rho(\alpha', \beta') = \rho_0 I_0(2\sqrt{\alpha' |\beta'|}) + \delta_0 \sqrt{\frac{|\beta'|}{\alpha'}}\, I_1(2\sqrt{\alpha' |\beta'|})$$

$$\delta(\alpha', \beta') = \delta_0 I_0(2\sqrt{\alpha' |\beta'|}) + \rho_0 \sqrt{\frac{\alpha'}{|\beta'|}}\, I_1(2\sqrt{\alpha' |\beta'|})$$

The radii of curvature of the α- and β-lines in the physical plane are

$$R(0, \beta) = R_0 + S_0 \beta, \qquad S(\gamma, 0) = S_0 + R_0 \gamma \tag{2.4}$$

$$R(\alpha, \beta) = R_0 I_0(2\sqrt{|\alpha| \beta}) + S_0 \sqrt{\frac{\beta}{|\alpha|}}\, I_1(2\sqrt{|\alpha| \beta})$$

$$S(\alpha, \beta) = S_0 I_0(2\sqrt{|\alpha| \beta}) + R_0 \sqrt{\frac{|\alpha|}{\beta}}\, I_1(2\sqrt{|\alpha| \beta})$$

The formulas (2.3) and (2.4) represent a generalization of the result obtained by Hill for a field bounded by arcs of equal radii.

Finally, the coordinates of the nodal points in the field of slip-lines formed by the circular arcs are

$$X(\gamma, 0) = R_0 \sin \gamma, \qquad X(0, \beta) = S_0 (1 - \cos \beta) \tag{2.5}$$

$$X(\alpha, \beta) = R_0\, U_1(2|\alpha|, 2i\sqrt{|\alpha| \beta}) + S_0 U_2(2\beta, 2i\sqrt{|\alpha| \beta})$$

$$Y(\gamma, 0) = R_0 (1 - \cos \gamma), \qquad Y(0, \beta) = S_0 \sin \beta$$

$$Y(\alpha, \beta) = R_0\, U_2(2|\alpha|, 2i\sqrt{|\alpha| \beta}) + S_0 U_1(2\beta, 2i\sqrt{|\alpha| \beta})$$

It must be noted that, since the function U_1 is odd with respect to the first argument, the formulas (2.5) are identical with the result obtained in [2].

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KÁRMÁN HYPOTHESES AND POWER LAWS FOR THE VARIATION OF ENERGY AND LINEAR SCALE OF TURBULENCE

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We consider the connection between the Kármán hypotheses concerning "self-preservation" (self-similarity) of correlation functions and power laws for the variation of the energy and the linear scale of turbulence downstream of a grid. Laws, based on the Kármán hypotheses, are close, in the intervals where measurements are made, to power laws (they argree with the experimental data at least as well as power laws) and they possess certain advantages from the point of view of the theory of homogeneous turbulence of a viscous incompressible fluid. We show that solutions, based on the Kármán hypotheses, are compatible with the equations for the higher moments, even if we assume for these moments additional hypotheses similar to the Kármán hypotheses.

1. A theory for the decay of a homogeneous isotropic turbulent motion of a viscous incompressible fluid can proceed from various hypotheses relating to the behavior of the velocity correlation moments of the second and third order, for example, the Kármán hypotheses [1]

$$b_d^{\,d}(r, t) = \langle u(0, t)\, u(r, t)\rangle = b(t) f(\chi) \qquad (1.1)$$

$$b_d^{\,nn}(r, t) = \langle u(0, t)\, v^2(r, t)\rangle = b^{3/2} h(\chi) \qquad (1.2)$$

$$(b(t) = b_d^{\,d}(0, t) = \langle u^2\rangle = \langle v^2\rangle, \quad \chi = r / l(t))$$

Here r is the distance, t is the time, u is the projection of the velocity pulsation in